Problem 1.

$$X_n = \max\left(X_{n-1}, \frac{X_{n-1}+U_n}{2}\right),$$

(a) Observe that $X_n \le 1$ a.s. since $X_1 = \frac{U_n}{2}$ and (By induction)

$$0 \le X_n \le \max\left(1, \frac{1+U_n}{2}\right) \le 1 \text{ a.s.}$$

Since the sequence X_n is increasing w.p. 1, we surmise that $X_n \xrightarrow{P} 1$. Let us show that this is the case.

$$P(X_n \le 1-\varepsilon) = P\left(X_{n-1} \le 1-\varepsilon, \frac{X_{n-1}+U_n}{2} \le 1-\varepsilon\right)$$

$$= P(X_{n-1} \le 1-\varepsilon, U_n \le 1-\varepsilon) = P(X_{n-1} \le 1-\varepsilon) P(U_n \le 1-\varepsilon)$$

$$= P(X_{n-1} \le 1-\varepsilon)(1-\varepsilon) = (1-\varepsilon)^n \to 0 \text{ as } n \to \infty$$

$\therefore\ X_n \xrightarrow{P} 1.$

(b) We have $P(X_n \le 1-\varepsilon) = (1-\varepsilon)^n$, and

$$\sum_{n=1}^{\infty} P(X_n \le 1-\varepsilon) = \sum_{n=1}^{\infty} (1-\varepsilon)^n = \frac{1-\varepsilon}{\varepsilon}$$

$\therefore\ P(X_n \le 1-\varepsilon \text{ i.o.}) = 0$, i.e., $X_n \xrightarrow{a.s.} 1$

(c) $$P((X_n-1)^2 < \varepsilon^2) = P(|X_n-1| < \varepsilon) = P(X_n > 1-\varepsilon) \xrightarrow[n\to\infty]{} 1$$

$$\lim_{n\to\infty} E(X_n-1)^2 \le \lim_{n\to\infty}\left(\underbrace{P((X_n-1)^2 > \varepsilon^2)}_{\downarrow 0} \cdot 1 + P((X_n-1)^2 < \varepsilon^2) \cdot \varepsilon^2\right)$$

$$\le \varepsilon^2$$

Since ε is arbitrarily small, this proves that $X_n \xrightarrow{L^2} 1$.

Problem 2.

(a) $$E\left[\sum_{j=1}^{k(n)}\left(B(t_j^{(n)})-B(t_{j-1}^{(n)})\right)^2-t\right]^2=E\left[\sum_{j=1}^{k(n)}\left(\left(B(t_j^{(n)})-B(t_{j-1}^{(n)})\right)^2-(t_{j+1}^{(n)}-t_j^{(n)})\right)\right]^2$$

by independence of increments

$$=\sum_{j=1}^{k(n)}E\left[\left(B(t_j^{(n)})-B(t_{j-1}^{(n)})\right)^2-(t_j^{(n)}-t_{j-1}^{(n)})\right]^2\leq\sum_{j=1}^{k(n)}\left[E\left(B(t_j^{(n)})-B(t_{j-1}^{(n)})\right)^4+(t_j^{(n)}-t_{j-1}^{(n)})^2\right]$$

$$\lesssim\sum_{j=1}^{k(n)}\left[c(t_j^{(n)}-t_{j-1}^{(n)})^4+(t_j^{(n)}-t_{j-1}^{(n)})^2\right]\leq\sum_{j=1}^{k(n)}c'(t_j^{(n)}-t_{j-1}^{(n)})\xrightarrow[n\to\infty]{}0\ ;\quad c,c'>0$$

(b) $$P\left(\left|\sum_{j=1}^{k(n)}\left(B(t_j^{(n)})-B(t_{j-1}^{(n)})\right)^2-t\right|>\varepsilon\right)\leq\varepsilon^{-2}E\left[\left(\sum_{j=1}^{k(n)}\left(B(t_j^{(n)})-B(t_{j-1}^{(n)})\right)^2-t\right)^2\right]$$

$$\leq\varepsilon^{-2}\cdot c'\sum_{j=1}^{k(n)}(t_j^{(n)}-t_{j-1}^{(n)})\quad\text{by part (a)}$$

If $$\sum_{n=1}^{\infty}\sum_{j=1}^{k(n)}(t_{j+1}-t_j)^2<\infty$$

then $$P\left(\left|\sum_{j=1}^{k(n)}\left(B(t_j^{(n)})-B(t_{j-1}^{(n)})\right)^2-t\right|>\varepsilon\ \text{i.o.}\right)\to 0$$

and therefore the convergence holds with prob. 1.

Problem 3.

(a) $G(s) = \sum_{n=0}^{\infty} P(Z=n)\, s^n = \sum_{n=0}^{\infty} p((1-p)s)^n = \dfrac{p}{1-(1-p)s}$, $|s| < \dfrac{1}{1-p}$

$$EZ = G'(1) = \frac{+p}{(1-(1-p)s)^2}\cdot(1-p)\Big|_{s=1} = \frac{p(1-p)}{p^2} = \frac{1-p}{p}.$$

If $\frac{1-p}{p} < 1$, i.e, $p > 1/2$, extinction prob. $= 1$

Otherwise, find s from $G(s) = s$: $\dfrac{p}{1-(1-p)s} = s$

giving $s = \dfrac{p}{1-p}$, as required

(b) $G(s) = \sum_{i=0}^{\infty} e^{-\lambda}\dfrac{(\lambda s)^i}{i!} = e^{-\lambda}\cdot e^{\lambda s} = e^{-\lambda(1-s)}$

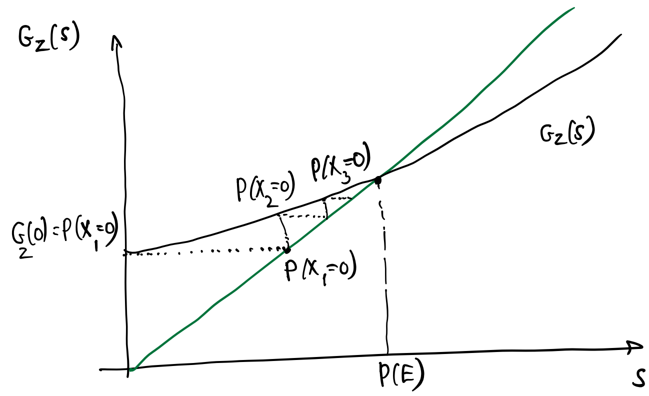
$$EZ = G'(1) = \lambda e^{-\lambda(1-s)}\Big|_{s=1} = \lambda.$$

If $\lambda = \ln 2 < 1$, the prob. of extinction $= 1$.

If $\lambda = \ln 4$, then the condition $e^{-\lambda(1-s)} = s$ yields

$-\ln 4\cdot(1-s) = \ln s$, or $P(E) = \frac{1}{2}$

(c)



We have $x_{n+1} = G_Z(x_n)$, $x_0 = 0$.

$$P(E) - x_{n+1} = P(E) - G_Z(x_n) = G_Z(P(E)) - G(x_n)$$

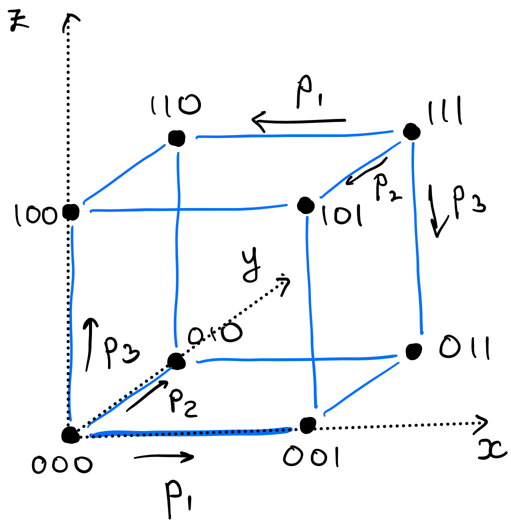
$$\frac{P(E) - x_{n+1}}{P(E) - x_n} = \frac{G_Z(P(E)) - G_Z(x_n)}{P(E) - x_n} \leq G_Z'(P(E)) \text{ by convexity of } G_Z.$$

Thus $P(E) - x_{n+1} \leq G_Z'(P(E))\,(P(E) - x_n) \leq \dots \leq (G_Z'(P(E)))^n$.

Note that $x_{n+1} = P(X_{n+1} = 0) = P(\tau \leq n)$, and $G_Z'(s) = \lambda G(s)$

$P(E) - P(\tau \leq n) \leq (\lambda P(E))^n$.

Problem 4.



(a) The transition matrix has the form

$$P = \begin{array}{c} \\ 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \begin{bmatrix} 0 & p_1 & p_2 & 0 & p_3 & 0 & 0 & 0 \\ p_1 & 0 & 0 & p_2 & 0 & p_3 & 0 & 0 \\ p_2 & 0 & 0 & p_1 & 0 & 0 & p_3 & 0 \\ 0 & p_2 & p_1 & 0 & 0 & 0 & 0 & p_3 \\ p_3 & 0 & 0 & 0 & 0 & p_1 & p_2 & 0 \\ 0 & p_3 & 0 & 0 & p_1 & 0 & 0 & p_2 \\ 0 & 0 & p_3 & 0 & p_2 & 0 & 0 & p_1 \\ 0 & 0 & 0 & p_3 & 0 & p_2 & p_1 & 0 \end{bmatrix} \end{array}$$

(b) Clearly if we take $\pi = \frac{1}{8}(1\,1\,1\,1\,1\,1\,1\,1)$, then $\pi P = \pi$ so π is a stationary distribution; at the same time, the process is periodic with period 2, so the limiting distribution does not exist.

(c) Write v_0 for vertex 000; write v_1 for vertex 111.

Let $A = \{$reaching v_1 without visiting v_0, starting at $v_0\}$

$B = \{$reaching v_0 without visiting v_1, starting at $v_1\}$

By symmetry, $P(A) = P(B)$

Denote this probability by s. Note that

$$P_{v_0}(\text{returning to } v_0 \text{ before hitting } v_1) = 1-s.$$

for the first time

Let $A_n = \{$returning to v_0 after exactly n visits to $v_1\}$, $n \geq 0$

$P(A_0) = 1-s$

$P(A_1) = P(A \cap B) = s^2(1-s)$ using the strong Markov property.

$\vdots$

$P(A_n) = s^2(1-s)^n$, $n \geq 1$

Let N be the number of times the walk visits v_1 before returning to v_0, starting at v_0.

$\{N = n\} = \{$hitting v_1 and returning to v_1 $(n-1)$ more times before finally returning to $v_0\}$

$$EN = \sum_{n\geq 1} s^2(1-s)^{n-1} n = s^2 \sum_{n\geq 1} (1-s)^{n-1} n = s^2 \left(-\sum_{n\geq 0} (1-s)^n\right)'_s$$

$$= s^2 \left(\frac{-1}{s}\right)'_s = 1.$$

Note that $s \neq 0, 1$ by our assumptions.

Problem 5.

(a) First, note that Y_k is a function of $X_1, \dots, X_k$, so it is measurable with respect to $\mathcal{F}_k$, so $(Y_k)_k$ forms an adapted system.
Plainly, Y_k is an L_1 random variable.

(b) Now

$$Y_{k+1} = \Phi\left(\frac{a-(S_k+X_{k+1})}{\sqrt{n-k-1}}\right)$$

$$E(Y_{k+1} \mid \mathcal{F}_k) = E\left(\Phi\left(\frac{a-S_k-X_{k+1}}{\sqrt{n-k-1}}\right) \Big| \mathcal{F}_k\right)$$

Note that $S_k \in \mathcal{F}_k$, so the expectation on the right is with respect to X_{k+1} which is independent of X_k. Thus, we have

$$E\,\Phi\left(\frac{a-S_k-X_{k+1}}{\sqrt{n-k-1}}\right) = \int_{-\infty}^{\infty} \Phi\left(\frac{a-S_k-x}{\sqrt{n-k-1}}\right)\varphi(x)\,dx, \text{ where } \varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad (1)$$

where we treat S_k as a constant.

Let $\xi \sim \mathcal{N}(\mu, \sigma^2)$, $z \in \mathbb{R}$. We have

$$P(\xi \le z) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{z} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy = \Phi\left(\frac{z-\mu}{\sigma}\right).$$

Let $\eta \sim \mathcal{N}(0,1)$, $\xi \perp\!\!\!\perp \eta$, then

$$P(\xi \le \eta) = E_\eta\, P(\xi \le z \mid \eta = z) = \int_{-\infty}^{\infty} \Phi\left(\frac{z-\mu}{\sigma}\right)\varphi(z)\,dz$$

which has the same form as (1).

Next, $\xi - \eta \sim \mathcal{N}(\mu, \sigma^2+1)$ by basic properties of independent Gaussian RVs, so

$$P(\xi \le \eta) = P(\xi - \eta \le 0) = \Phi\left(\frac{-\mu}{\sqrt{1+\sigma^2}}\right).$$

Use this relation in (1):

$$E\left(\Phi\left(\frac{a-S_k-X_{k+1}}{\sqrt{n-k-1}}\right) \Big| \mathcal{F}_k\right) = \Phi\left(\frac{a-S_k}{\sqrt{n-k}}\right) = Y_k, \text{ finishing the proof.}$$